On the Physical Meaning of the Curl Operator

by Christopher K. Horne

In solving electromagnetic problems where the curl operator is evoked to compute the electric or magnetic fields, one often forgets the curl has a physical meaning. The purpose of this article is to support the mathematics of the curl with the physical meaning of its operation.

James C. Maxwell first coined the name curl[1] from a popular name found in his native-born country of Scotland. Maxwell often referred to the curl operator and its vector equations as "the equation of magnetic force". The curl operator differs from the divergence operator in that it acts on a vector field to produce another vector field. Since vectors are more complicated than scalars it is important to understand the purpose of the curl operator.

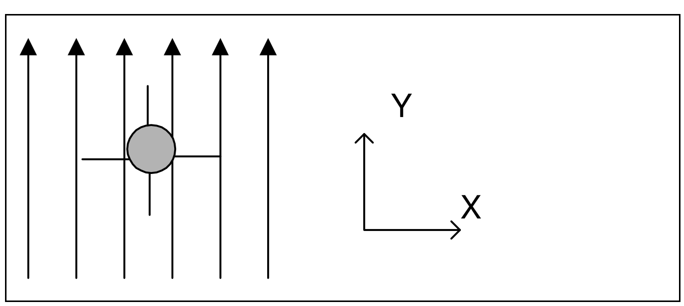
Let $\vec{A}(x, y, z)$ be any vector (i.e. has magnitude and position). The mathematical expression for the curl of $\vec{A}$ is written as

$$\nabla\times\vec{A} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial Az}{\partial y} - \frac{\partial Ay}{\partial z}\right)\vec{e}_x + \left(\frac{\partial Ax}{\partial z} - \frac{\partial Az}{\partial x}\right)\vec{e}_y + \left(\frac{\partial Ay}{\partial x} - \frac{\partial Ax}{\partial y}\right)\vec{e}_z \qquad \text{(Eqn. 1)}$$

The physical significance of the curl operator is that it describes the "rotation" of the field $\vec{A}$ at a point in question. Imagine a pipe or stream of flowing water, such that the velocity of the flow at any point (x, y, and z) is equal to $\vec{A}(x, y, z)$. Now imagine that we insert a shaft with a paddle wheel on the end. The shaft points in the z-direction and measures its rotation or "curl" as a z-component.

If there is no rotation in the water stream the wheel will not turn (see Figure 1 below). If the wheel does not turn, we see there is no change in x –direction and the partial derivative, $\frac{\partial A_y}{\partial x} = 0$. Also, there are equal forces in the z-direction and thus $\nabla\times A_z = 0$.

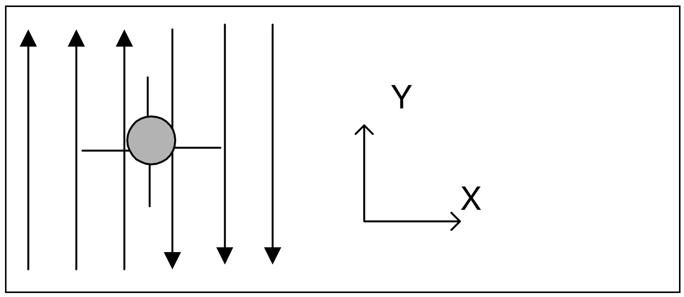
Figure 1: In a field with equal strength on both sides of the paddle wheel (shaded circle pointing into page), the z-directed curl is zero.



[1] "Maxwell on the Electromagnetic Field" by Thomas K. Simpson, page 201

Now suppose one side of the stream has more intense flow. If the current in the stream is swifter on one side than the other, then the paddle wheel will turn. The rate of change in the x-direction will not be zero anymore (i.e. $\frac{\partial A_y}{\partial x} \neq 0$). An illustration of this is shown in Figure 2 below.

Figure 2: Vector field $\overrightarrow{A_y}$ where there is "rotation" ; the curl is non-zero.

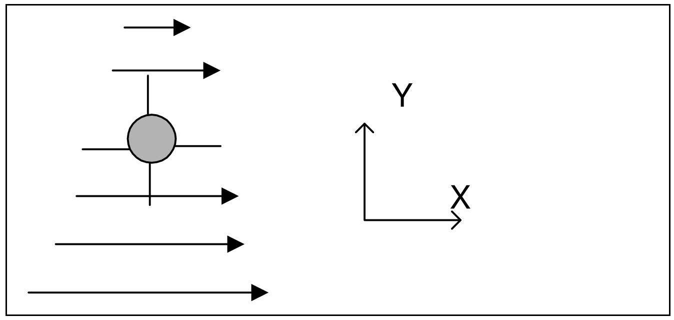


In this case, the field is changing at the paddle wheel and $\frac{\partial Ay}{\partial x} > 0$. From Equation 1, we have

$$\nabla \times A_z = \frac{\partial Ay}{\partial x} \vec{e}_z$$

Now consider a vector field whose magnitude diminishes as a function of distance y. This field is shown pictorially in Figure 3 (next page). In this case $\frac{\partial A_x}{\partial y} < 0$ and the force on the lower side of the wheel is greater than the force on the top; the paddle wheel turns counter-clockwise. For this case, there is rotation on the z-directed shaft, and so the curl of $\vec{A}_z$, $\nabla \times A_z = \frac{\partial Ax}{\partial y} \vec{e}_z < 0$

Figure 3: Vector field $\overrightarrow{A_y}$ whose magnitude diminishes as a function of y ; the curl $\nabla \times A_z < 0$; the paddle turns counter-clockwise.



In Summary, one finds the curl of any vector field by simply applying Equation 1 for the Cartesian system or an appropriate formula for the cylindrical or spherical systems. However, care must be taken in order to determine the correct results and understanding of the physical meaning will provide deeper insight into the problem.